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Separating Classical and Quantum Correlations[1]

E. G. Beltrametti[2,3,6] and S. Bugajski[4,5]

We examine to what extent the correlation between two quantum observables at a mixed state can be separated into a classical and a quantum term. The nonunique decomposition of quantum mixed states into pure states makes such a separation ambiguous. We outline this fact by a simple example, which also shows that classical and quantum correlations may cancel each other out.

KEY WORDS: quantum mechanics; correlations; entanglement; mixed states.

1. INTRODUCTION

The notion of correlation between two observables always refers, explicitly or implicitly, to a joint observable, hence to a joint measurement. A new approach to characterize classical and quantum correlations, based on probability theory, has been recently proposed in Beltrametti and Bugajski (2002, 2003); the description of both kinds of correlation can be properly done in terms of density functions. In Section 2 a partial account of this approach is given.

Although we are keen to believe that the concept of quantum correlation captures the quantum phenomenon of *entanglement*, we will not refer to the latter term, in view of different and presumably inequivalent meanings it acquired in the literature.

In the classical case the notion of correlation always calls into play a mixed state of the physical system: every correlation disappears at pure states. Thus the classical correlation has its roots in the way the pure states are mixed up to form the actual (mixed) state of the physical system. On the contrary, the quantum frame can give rise to correlations that persist at pure states, so that we might figure two

[1] This paper was written a few months before the death of S. Bugajski: the first author recalls him as a creative scientist, a great human personality, and a dear friend.

[2] Department of Physics, University of Genoa, Italy.
[3] Istituto Nazionale di Fisica Nucleare, Sezione di Genova, Italy.
[4] Institute of Physics, University of Silesia, Poland.
[5] Institute of Pure and Applied Informatics, Polish Academy of Sciences, Poland.
[6] To whom correspondence should be addressed at Department of Physics, University of Genoa, via Dodecaneso 33, 16146 Genova, Italy; e-mail: beltrametti@ge.infn.it.

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different roots of a correlation between quantum observables: a typical quantum correlation sitting at the pure states and an additional correlation having a classical nature, generated by the mixing of pure states. In this paper we examine to what extent these two kinds of correlations can be consistently separated within the standard framework of quantum mechanics. We will see that such a separation faces ambiguities when we deal with quantum mixed states due to their nonunique decomposition into pure states; in fact, a quantum mixed state is represented by a density operator (of a Hilbert space) and the latter is known to admit infinitely many convex decompositions into one-dimensional projectors, i.e., into pure states.

These features will be made explicit by adopting the simple example of Section 3, i.e., a physical system composed of two spin-$\frac{1}{2}$ subsystems. In Section 4 we exhibit how the separation of the correlation into a classical and a typical quantum term depends on the statistical content of the mixed state, namely on a piece of information which is not unambiguously coded in the standard quantum frame. The unexpected fact that the classical and the quantum term may combine to produce no total correlation is outlined in Section 5.

The issue of characterizing classical and quantum correlations has been discussed in the literature under various perspectives: we quote Horodecki *et al.* (2001), Henderson and Vedral (2001), Keyl (2002), and Majewski (2002) for further bibliography.

2. CLASSICAL AND QUANTUM CORRELATIONS

Consider two observables A_1, A_2, and denote Ξ_1, Ξ_2 the measurable spaces in which they take values. To a state α of the physical system under discussion, the two observables will associate the probability measures $P(A_1, \alpha)$ and $P(A_2, \alpha)$ on Ξ_1 and Ξ_2, respectively: if $X_i \subseteq \Xi_i, i = 1, 2$, then $P(A_i, \alpha)(X_i)$ will represent the probability that A_i takes a value in X_i at the state α. Let $A_{1,2}$ denote the joint observable we are referring to: to the state α it will associate the probability measure $P(A_{1,2}, \alpha)$ on the Cartesian product $\Xi_1 \times \Xi_2$. According to the probability theory the two observables A_1, A_2 are said to be correlated at the state α, relative to the joint observable $A_{1,2}$, if $P(A_{1,2}, \alpha)$ differs from the product of the two measures $P(A_1, \alpha)$ and $P(A_2, \alpha)$, namely if

$$P(A_{1,2}, \alpha) \neq P(A_1, \alpha) \boxtimes P(A_2, \alpha),$$

where $P(A_1, \alpha) \boxtimes P(A_2, \alpha)$ is the measure on $\Xi_1 \times \Xi_2$ defined by $P(A_1, \alpha) \boxtimes P(A_2, \alpha)\,(X_1 \times X_2) = P(A_1, \alpha)(X_1) \cdot P(A_2, \alpha)(X_2)$ for every $X_1 \subseteq \Xi_1, X_2 \subseteq \Xi_2$.

In this paper Ξ_1 and Ξ_2 will refer to discrete subsets of the reals: if $\xi_1 \in \Xi_1$, $\xi_2 \in \Xi_2$, the correlation between A_1 and A_2 at the state α, relative to the joint observable $A_{1,2}$, will then be completely encoded in the real-valued function ρ on

$\Xi_1 \times \Xi_2$ defined by

$$\rho(\xi_1, \xi_2) := \frac{P(A_{1,2}, \alpha)(\xi_1, \xi_2)}{P(A_1, \alpha)(\xi_1) \cdot P(A_2, \alpha)(\xi_2)}, \tag{1}$$

and called the correlation density function (actually the Radon-Nicodym derivative, see for instance Bauer (1981) of $P(A_{1,2}, \alpha)$ with respect to $P(A_1, \alpha) \boxtimes P(A_2, \alpha)$.

In the sequel we will mainly refer to the case in which α is a mixed state and we write

$$\alpha = \sum_n w_n P_n, \quad 0 \leq w_n \leq 1, \quad \sum_n w_n = 1, \tag{2}$$

for its convex decomposition into the family $\{p_n\}$ of pure states (for simplicity we restrict to discrete convex combinations). Since the physical observables preserve the convex structure, we have, for an arbitrary observable A,

$$P(A, \alpha) = \sum_n w_n p(A, p_n).$$

In the classical case, the convex decomposition of mixed state into pure states is known to be unique; in other words, the set of all states has the peculiar convex structure of a simplex. Moreover, for any two observables A_1, A_2 there is a unique joint observable defined by

$$P(A_{1,2}, \alpha) = \sum_n w_n P(A_1, p_n) \boxtimes P(A_2, p_n), \tag{3}$$

so that the correlation density function takes the classical form

$$\rho_c(\xi_1, \xi_2) := \frac{\sum_n w_n P(A_1, p_n)(\xi_1) \cdot P(A_2, p_n)(\xi_2)}{P(A_1, \alpha)(\xi_1) \cdot P(A_2, \alpha)(\xi_2)}. \tag{4}$$

The distinguishing feature of the classical correlation is that the density function ρ_c equals 1 at every pure state: any two observables show no correlation (hence they are independent) at every pure state. This fact supports the statement that the classical correlation has its roots in the way the pure states are mixed together to form the physical state under discussion.

Things are different in the quantum case. The convex decomposition of a mixed state into pure states is no longer unique, and two observables A_1, A_2 need not admit a joint observable. Only in cases A_1 and A_2 are compatible (namely when they are represented by commuting operators) is the existence, as well as the unicity, of the joint observable $A_{1,2}$ ensured, though the expression of Eq. (3) need not apply. Of course the correlation between A_1 and A_2 at the state α, relative to the joint observable $A_{1,2}$, is still fully specified by a density function as in Eq. (1). The peculiar feature of the quantum context is that two observables can exhibit a correlation even at a pure state: the correlation density function need not be the constant unit function at the pure states.

Thus, in the quantum case the correlation between two observables appears to encompass two possible faces: a correlation already present at the level of pure states and a correlation produced, as in the classical case, by the mixing of the pure states occurring in the mixed state we are dealing with. The natural question then arises whether the correlation between two quantum observables may be separated into a classical term and a typical quantum term. As discussed in more detail in Beltrametti and Bugajski (2002, 2003) a formal answer to such a question corresponds to rewrite the density function ρ, as defined in Eq. (1), in the factorized form

$$\rho = \rho_c \cdot \rho_q, \tag{5}$$

where ρ_c is given in Eq. (4), and ρ_q is defined by

$$\rho_q(\xi_1, \xi_2) = \frac{P(A_{1,2}, \alpha)(\xi_1, \xi_2)}{\sum_n w_n P(A_1, p_n)(\xi_1) \cdot P(A_2, p_n)(\xi_2)}. \tag{6}$$

We are however faced with a problem: the nonuniquesess of the convex decomposition of a quantum mixed state into pure states makes the quantity

$$\sum_n w_n P(A_1, p_n)(\xi_1) \cdot P(A_2, p_n)(\xi_2) \tag{7}$$

occurring in the numerator of Eq. (4) and in the denominator of Eq. (6), crucially dependent upon the particular convex combination we adopt for the mixed state under discussion. Hence, also the two factors ρ_c and ρ_q in Eq. (5) crucially depend upon the convex decomposition of the mixed state we refer to. The physical system described in the next section will offer a simple check of all that.

3. A TWO SPIN-$\frac{1}{2}$ SYSTEM

Consider a two spin-$\frac{1}{2}$ system described in terms of the tensor product space $\mathbf{C}^2 \otimes \mathbf{C}^2$. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be orthonormal vectors of $\mathbf{C}^2$, and take the two comeasurable "local" spin observables (with some abuse of notation we use the same symbol for an observable and for the corresponding operator)

$$A_1 := s_z \otimes I, \quad A_2 := I \otimes s_z,$$

where

$$s_x := \frac{1}{2}|\uparrow\rangle\langle\uparrow| - \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

is the one-particle operator representing the z-component of the spin.

Both A_1 and A_2 take values in the set $\Xi_1 = \Xi_2 = \{\frac{1}{2}, -\frac{1}{2}\}$ and they correspond to projection-valued measures (PV measures) on this set, to be denoted E^{A_1} and

E^{A_2}. Clearly we have

$$E^{A_1}\left(\frac{1}{2}\right) = |\uparrow\rangle\langle\uparrow| \otimes I, \quad E^{A_1}\left(-\frac{1}{2}\right) = |\downarrow\rangle\langle\downarrow| \otimes I,$$

$$E^{A_2}\left(\frac{1}{2}\right) = I \otimes |\uparrow\rangle\langle\uparrow|, \quad E^{A_2}\left(-\frac{1}{2}\right) = I \otimes |\downarrow\rangle\langle\downarrow|.$$

A state of this physical system will be represented by a density operator D of $\mathbf{C}^2 \otimes \mathbf{C}^2$, which becomes a one-dimensional projector if we deal with a pure state. According to basic rules of quantum mechanics the observables A_1, A_2 determine, at the state D, the probability measures on $\{\frac{1}{2}, -\frac{1}{2}\}$:

$$P(A_1, D) = \mathrm{Tr}(|\uparrow\rangle\langle\uparrow| \otimes I \cdot D)\eta_{\frac{1}{2}} + \mathrm{Tr}(|\downarrow\rangle\langle\downarrow| \otimes I \cdot D)\eta_{-\frac{1}{2}}, \tag{8}$$

$$P(A_2, D) = \mathrm{Tr}(I \otimes |\uparrow\rangle\langle\uparrow| \cdot D)\eta_{\frac{1}{2}} + \mathrm{Tr}(I \otimes |\downarrow\rangle\langle\downarrow| \cdot D)\eta_{-\frac{1}{2}}. \tag{9}$$

Here $\eta_{\frac{1}{2}}$ and $\eta_{-\frac{1}{2}}$ denote the measures on $\{\frac{1}{2}, -\frac{1}{2}\}$ concentrated, respectively, at the point $\{\frac{1}{2}\}$ and at the point $\{-\frac{1}{2}\}$.

The joint measurement of these observables is just a measurement of their joint observable $A_{1,2}$. In the quantum framework the latter is uniquely specified by the PV measure $E^{A_{1,2}}$ on the Cartesian product $\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\}$ which takes the values

$$E^{A_{1,2}}\left(\frac{1}{2}, \frac{1}{2}\right) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow|, \quad E^{A_{1,2}}\left(\frac{1}{2}, -\frac{1}{2}\right) = |\uparrow\downarrow\rangle\langle\uparrow\downarrow|,$$

$$E^{A_{1,2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = |\downarrow\uparrow\rangle\langle\downarrow\uparrow|, \quad E^{A_{1,2}}\left(-\frac{1}{2}, -\frac{1}{2}\right) = |\downarrow\downarrow\rangle\langle\downarrow\downarrow|,$$

where $|\uparrow\uparrow\rangle$ is an abbreviation for $|\uparrow\rangle \otimes |\uparrow\rangle$, and similarly for $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$.

The probability measure on $\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\}$ determined by $A_{1,2}$ at D will then become

$$\begin{aligned} P(A_{1,2}, D) = {} & \mathrm{Tr}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| \cdot D)\eta_{(\frac{1}{2},\frac{1}{2})} + \mathrm{Tr}(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| \cdot D)\eta_{(\frac{1}{2},-\frac{1}{2})} \\ & + \mathrm{Tr}(|\downarrow\uparrow\rangle\langle\downarrow\uparrow| \cdot D)\eta_{(-\frac{1}{2},\frac{1}{2})} + \mathrm{Tr}(|\downarrow\downarrow\rangle\langle\downarrow\downarrow| \cdot D)\eta_{(-\frac{1}{2},-\frac{1}{2})} \end{aligned} \tag{10}$$

where $\eta_{(\frac{1}{2},\frac{1}{2})}$ is the measure concentrated at the point $\{\frac{1}{2}, \frac{1}{2}\}$ and similarly for $\eta_{(\frac{1}{2},-\frac{1}{2})}$, $\eta_{(-\frac{1}{2},\frac{1}{2})}$, and $\eta_{(-\frac{1}{2},-\frac{1}{2})}$.

4. THE ROLE OF THE STATISTICAL CONTENT OF A STATE

As already recalled, a quantum mixed state admits infinitely many decompositions into pure states. By *statistical content* of a quantum mixed state we mean a particular decomposition of the corresponding density operator into one-dimensional projectors. We will show that the statistical content of a state is crucial for separating classical and quantum correlations.

With reference to the physical system of the previous section, consider the mixed state

$$D = w(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) + w'(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|) \qquad (11)$$

where w, w' are positive numbers such that $w + w' = \frac{1}{2}$. The right hand side of this equation represents a particular statistical content of the state.

The probability measures of Eqs. (8), (9), and (10) are easily seen to become, at the above state,

$$P(A_1, D) = P(A_2, D) = \frac{1}{2}\eta_{\frac{1}{2}} + \frac{1}{2}\eta_{-\frac{1}{2}}, \qquad (12)$$

$$P(A_{1,2}, D) = w\eta_{(\frac{1}{2},\frac{1}{2})} + w'\eta_{(\frac{1}{2},-\frac{1}{2})} + w'\eta_{(-\frac{1}{2},\frac{1}{2})} + w\eta_{(-\frac{1}{2},-\frac{1}{2})}. \qquad (13)$$

Hence, the density function of the correlation between A_1 and A_2 at D reads, according to Eq. (1),

$$\rho\left(\frac{1}{2},\frac{1}{2}\right) = \rho\left(-\frac{1}{2},-\frac{1}{2}\right) = 4w, \quad \rho\left(\frac{1}{2},-\frac{1}{2}\right) = \rho\left(-\frac{1}{2},\frac{1}{2}\right) = 4w'. \quad (14)$$

In order to discuss the separation of this correlation into a classical and a quantum term we have to take, as discussed in Section 2, the probability measure of Eq. (7), and refer it to the convex decomposition in Eq. (11). By inspection of Eq. (12) we obtain a probability measure which reproduces exactly $P(A_{1,2}, D)$ of Eq. (13). Thus we see that the correlation between A_1 and A_2 at the state D appears to be entirely of a classical nature: with reference to Eqs. (4), (6) we have now $\rho = \rho_c$, while ρ_q becomes the constant unit function.

To show that the above result crucially depends upon the particular convex decomposition of Eq. (11), let us observe that the density operator D admits, among others, also the following decomposition into the so-called Bell base:

$$D = w(|B_1\rangle\langle B_1| + |B_2\rangle\langle B_2|) + w'(|B_3\rangle\langle B_3| + |B_4\rangle\langle B_4|), \qquad (15)$$

where

$$|B_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \quad |B_2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle),$$

$$|B_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |B_4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

A simple check proves that the probability measures $P(A_1, D)$, $P(A_2, D)$ are again as in Eq. (12), and $P(A_{1,2}, D)$ is again as in Eq. (13). This is an expected fact since the observables act affinely on the states, so that the probability measures on their outcome spaces do not depend on the specific statistical content of the state. Thus, the correlation between A_1 and A_2 at the state of Eq. (15) is again the one of Eq. (14).

But let us now look at the separation of this correlation into a classical and a quantum term. Again we have to take the probability measure of Eq. (7), referring now to the convex decomposition in Eq. (15): the result is a probability measure which coincides with $P(A_1, D) \boxtimes P(A_2, D)$, namely with the probability measure in which the two observables A_1 and A_2 behave as independent. This shows that, if we refer to the statistical content of D expressed by Eq. (15), the correlation between A_1 and A_2 appears to be entirely of a quantum nature (*entanglement*), without any classical correlation coming into play. In other words we have now $\rho = \rho_q$, while ρ_c becomes the constant unit function.

Thus, we see that the correlation between two observables at a given mixed state may appear purely classical or purely quantum depending on the statistical content we refer to.

5. A CANCELING OUT EFFECT

The said dependence on the statistical content of the state can be further emphasized by noticing that the quantum mixed state D of Eq. (11) or of Eq. (15) admits also the alternative convex decomposition

$$D = w(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) + w'(|B_3\rangle\langle B_3| + |B_4\rangle\langle B_4|). \tag{16}$$

As expected, and easily checked, even referring to this new statistical content of D we get again the probability measures $P(A_1, D)$, $P(A_2, D)$, and $P(A_{1,2}, D)$ of Eqs. (12) and (13). Hence we find again the correlation between A_1 and A_2 as expressed by Eq. (14). To examine the separation of this correlation into a classical and a quantum part we have now to consider the probability measure of Eq. (7) and make reference to the convex decomposition of Eq. (16): it is easily seen that we come to the probability measure

$$\frac{2w + w'}{2}\eta_{(\frac{1}{2},\frac{1}{2})} + \frac{w'}{2}\eta_{(\frac{1}{2},-\frac{1}{2})} + \frac{w'}{2}\eta_{(-\frac{1}{2},\frac{1}{2})} + \frac{2w + w'}{2}\eta_{(-\frac{1}{2},-\frac{1}{2})}.$$

Since this probability measure does not coincide neither with $P(A_{1,2}, D)$ nor with $P(A_1, D) \boxtimes P(A_2, D)$ we have to conclude that the correlation between A_1 and A_2 exhibits now both a classical and a quantum term. With reference to Eqs. (4) and (6), and by inspection of Eqs. (12) and (13), we get indeed

$$\begin{aligned} \rho_c\left(\frac{1}{2}, \frac{1}{2}\right) &= \rho_c\left(-\frac{1}{2}, -\frac{1}{2}\right) = 2(2w + w'), \\ \rho_c\left(\frac{1}{2}, -\frac{1}{2}\right) &= \rho_c\left(-\frac{1}{2}, \frac{1}{2}\right) = 2w', \end{aligned} \tag{17}$$

$$\rho_q\left(\frac{1}{2},\frac{1}{2}\right) = \rho_q\left(-\frac{1}{2},-\frac{1}{2}\right) = \frac{2w}{2w+w'},$$
$$\rho_q\left(\frac{1}{2},-\frac{1}{2}\right) = \rho_q\left(-\frac{1}{2},\frac{1}{2}\right) = 2. \tag{18}$$

Intuitively, the occurrence of both a classical and a quantum correlation can be traced back to the fact that we are now looking at a statistical content of the state D which involves product states in the subspace spanned by $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and Bell states in the subspace spanned by $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$: at product states only classical correlations appear, while at Bell states only quantum correlations do.

A puzzling canceling out effect emerges when we refer to the particular case $w = w' = \frac{1}{4}$. Indeed, the density function ρ (see Eq. (14)) becomes the constant unit function, while it is evident that neither ρ_c (see Eq. (17)) nor ρ_q (see Eq. (18)) becomes a constant function: in other words, the two observables A_1 and A_2 appear to be classically as well as quantum correlated though there is no total correlation. The classical and the quantum correlations cancel each other out—an effect that involves a sort of "hidden entanglement" and might deserve further attention.

Summing up, a consistent separation of a total correlation into a classical and a quantum term requires the knowledge of the particular statistical content of the mixed state we are dealing with, a knowledge sitting outside the standard quantum frame. This might appear unsatisfactory in view of the naturalness of the notions of classical and quantum correlations referred to. Let us notice that classical and quantum correlations can be consistently separated within the probability frame considered in Beltrametti and Bugajski (1995, 2002, 2003) and Bugajski (1996), where the adopted family of observables is rich enough to separate distinct statistical contents of a mixed state.

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